SPECTRAL MULTIPLICITIES FOR ERGODIC FLOWS

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Abstract. Let E be a subset of positive integers such that $E\cap\{1,2\}\neq\emptyset$. A weakly mixing finite measure preserving flow $T=(T_t)_{t\in\mathbb{R}}$ is constructed such that the set of spectral multiplicities (of the corresponding Koopman unitary representation generated by T) is E. Moreover, for each non-zero $t\in\mathbb{R}$, the set of spectral multiplicities of the transformation T_t is also E. These results are partly extended to actions of some other locally compact second countable Abelian groups.

0. Introduction

Let G be a locally compact second countable Abelian group and let $T=(T_g)_{g\in G}$ be a measure preserving action of G on a standard probability space $(X,\mathfrak{B},\mu)$. The spectral theory of dynamical systems studies the corresponding Koopman unitary representation $U_T=(U_T(g))_{g\in G}$ in the Hilbert space $L^2_0(X,\mu):=L^2(X,\mu)\ominus\mathbb{C}$ given by

$$U_T(g)f:=f\circ T_{-g}$$

(see [KaT]). Such a representation is completely characterized (up to unitary equivalence) by a measure of maximal spectral type on the dual group $\widehat{G}$ and a spectral multiplicity function $l_T:\widehat{G}\ni w\to l_T(w)\in\mathbb{N}\cup\{\infty\}$. We denote by $\mathcal{M}(T)=\mathcal{M}(U_T)$ the (essential) image of l_T.

One of the most appealing open problems in the spectral theory of dynamical systems can be stated as follows: when a unitary representation is unitarily equivalent to a Koopman representation? Let us consider a weak version of this problem by replacing the unitary equivalence with another (weaker) equivalence relation on the set of unitary representations of G. It was introduced in [Fr] for the unitary representations of $\mathbb{Z}$. Two unitary representations U and V of G in Hilbert spaces $\mathcal{H}_U$ and $\mathcal{H}_V$ respectively are called *cyclicly isomorphic* if there is a unitary operator $W:\mathcal{H}_U\to\mathcal{H}_V$ such that the image under W of each U-cyclic subspace in $\mathcal{H}_U$ is a V-cyclic subspace in $\mathcal{H}_V$ and vise versa. Based on the proof in [Fr] for $G=\mathbb{Z}$ it is easy to see that if U and V have a continuous spectrum then they are cyclicly isomorphic if and only if $\mathcal{M}(U)=\mathcal{M}(V)$. We thus come to the following natural question which is called the spectral multiplicity problem:

which subsets $E\subset\mathbb{N}$ are realizable as $E=\mathcal{M}(T)$ for an ergodic (or weakly mixing) free action T?

Research of the second named author was partially supported by Polish MNiSzW grant N N201 384834.

In the case $G = \mathbb{Z}$ the spectral multiplicity problem was studied by a number of authors (see references in a recent survey [Le] and subsequent progress in [Ry2], [KaL], [Da3]). It is proved, in particular, that a subset $E \subset \mathbb{N}$ is realizable if one of the following is fulfilled: $1 \in E$, $2 \in E$ or $E = n \cdot F$ for some $n \geq 2$ and a subset $F \subset \mathbb{N}$ with $1 \in F$. It is believed that every subset of $\mathbb{N}$ is realizable. In the case $G = \mathbb{Z}^2$, weakly mixing realizations of the subsets $E \ni 1$ were constructed in [Fi] and weakly mixing realizations of subsets $\{2, 4, \dots, 2^n\}$, for each $n > 0$, were constructed in [Ko]. For a large class of Abelian locally compact second countable groups G including all countable groups and $\mathbb{R}^n$, it was proved in [DaS] that there exist weakly mixing G-actions with homogeneous spectrum of arbitrary multiplicity.

In the present paper we mainly consider the case when $G = \mathbb{R}$.

Theorem 0.1 (Main theorem). *Let E be a subset of positive integers such that $E \cap \{1, 2\} \neq \emptyset$.*

(i) *There is a weakly mixing finite measure preserving flow $T = (T_t)_{t \in \mathbb{R}}$ such that the set of spectral multiplicities of the Koopman unitary representation generated by T is E.*
(ii) *For each non-zero $t \in \mathbb{R}$, the set of spectral multiplicities of the Koopman operator generated by the transformation T_t is also E.*

Now Theorem 0.1(i) can be interpreted in the following way: every unitary representation of $\mathbb{R}$ with continuous spectrum and such that $\mathcal{M}(U) \cap \{1, 2\} \neq \emptyset$ is cyclicly isomorphic to a Koopman representation of $\mathbb{R}$. Secondly, given a subset $E \subset \mathbb{N}$ such that $E \cap \{1, 2\} \neq \emptyset$, denote by $\mathcal{W}_E$ the set of weakly mixing transformations S with $\mathcal{M}(S) = E$. As was mentioned above, the set $\mathcal{W}_E$ is known to be non-empty. Theorem 0.1(ii) strengthens this fact: $\mathcal{W}_E \cup \{\mathrm{Id}\}$ contains a one-parameter subgroup.

Now we make some remarks concerning the proof of Theorem 0.1. The simplest way to obtain flows with non-trivial spectral properties is to consider the suspensions of ergodic transformations with non-trivial spectral properties. We recall that the suspensions are special flows constructed under the constant function 1. In other words, they are $\mathbb{R}$-actions induced by $\mathbb{Z}$-actions. In Section 1 we briefly review properties of induced actions in a more general setting of pairs (G, H), where G is a locally compact second countable Abelian group and H a closed co-compact subgroup of G. Some of these properties were established in the original papers by G. Mackey [Ma] and R. Zimmer [Zi]. By means of the inducing we can obtain "cheaply" a realization of each subset $E \subset \mathbb{N}$ containing 1 as the set of spectral multiplicities of an ergodic flow. Unfortunately, the condition $1 \in E$ is unavoidable within the class of suspension flows. Moreover, every suspension flow has a non-trivial discrete spectrum. Therefore to construct weakly mixing realizations we apply another approach. It is a continuous analogue of the realizations produced in [Da3]. The desired flows are compact group extensions of either rank-one flows (for realizations of sets $E \ni 1$ in Section 4) or Cartesian squares of rank-one flows (for realizations of sets $E \ni 2$ in Section 5). Sections 2 and 3 contain some preliminary material to understand the techniques used in Sections 4 and 5. In the final Section 6 we partly extend Theorem 0.1 to actions of other non-compact Abelian groups: connected groups, groups without non-trivial compact subgroups, groups containing a closed one-parameter subgroup, etc.

1. Induced actions

Let G be a locally compact second countable Abelian group and H a co-compact subgroup of G. Given a measure preserving action $S = (S_h)_{h\in H}$ of H on a standard probability space $(X, \mathfrak{B}, \mu)$, we denote by $T = (T_g)_{g\in G}$ the induced action of G on the product space $(G/H \times X, \lambda_{G/H} \times \mu)$, where $\lambda_{G/H}$ is Haar measure on G/H (see [Ma], [Zi]). Fix a Borel cross-section $s : G/H \to G$ of the natural projection $G \to G/H$ such that $s(H) = 0$. Then

$$T_g(y, x) := (gy, S_{h(g,y)}x), \tag{1-1}$$

where $h(g, y) := -s(gy) + g + s(y) \in H$. Notice that the mapping $h : G \times Y \to H$ is a 1-cocycle, i.e.

$$h(g_1g_2, y) = h(g_1, g_2y) + h(g_2, y)$$

for all $y \in Y$, $g_1, g_2 \in G$. If S is ergodic then so is T. Denote by U_T and U_S the Koopman representations of G and H generated by T and S respectively. Then U_T is unitarily equivalent to the unitary representation of G induced by U_S [Ma]. Recall that given a unitary representation $V = (V_h)_{h\in H}$ of H in a Hilbert space $\mathcal{H}$, the *induced (by V) representation* $U = (U_g)_{g\in G}$ of G is defined on the Hilbert space $L^2(G/H, \lambda_{G/H}) \otimes \mathcal{H}$ by the formula

$$U_{-g}f(y) := V_{h(g,y)}(f(gy)).$$

Here we consider $f \in L^2(G/H, \lambda_{G/H}) \otimes \mathcal{H}$ as a measurable function $f : G/H \to \mathcal{H}$ such that $\int_{G/H} \|f(y)\|^2\, d\lambda_{G/H}(y) < \infty$. In particular, under the above identification, if $b \in L^2(G/H, \lambda_{G/H})$ and $a \in \mathcal{H}$ then for $f(y) = (b \otimes a)(y) = b(y)a$ and $h \in H$ we obtain

$$(U_hf)(y) = b(y)V_h(a) = (b \otimes V_ha)(y).$$

Proposition 1.1. *Let $\pi : \widehat{G} \to \widehat{H}$ stand for the natural projection. Denote by σ_U a measure of maximal spectral type of U on $\widehat{G}$.*

(i) *$\sigma_U \circ \pi^{-1}$ is a measure of maximal spectral type of V.*
(ii) *U and V have the same set of spectral multiplicities.*

Proof. Let $M \subset \mathbb{N} \cup \{\infty\}$ denote the set of spectral multiplicities of V. Then there is a decomposition $\mathcal{H} = \bigoplus_{i\in M} \bigoplus_{j=1}^{i} \mathcal{H}_{i,j}$ of $\mathcal{H}$ such that

- $\mathcal{H}_{i,j}$ is a cyclic space for V for every pair (i, j). Denote by $\sigma_{i,j}$ a measure of the maximal spectral type for $V \restriction \mathcal{H}_{i,j}$. Then
- $\sigma_{i,j} \perp \sigma_{i',j'}$ if $i \neq i'$ and
- $\sigma_{i,j} \sim \sigma_{i',j'}$ if $i = i'$.

It is easy to see that $L^2(G/H, \lambda_{G/H}) \otimes \mathcal{H}_{i,j}$ is a cyclic space for U. Denote by $\sigma'_{i,j}$ a measure of the maximal spectral type of $U \restriction (L^2(G/H, \lambda_{G/H}) \otimes \mathcal{H}_{i,j})$. It is easy to see that

$$\sigma'_{i,j} \sim \sigma'_{i',j'} \quad \text{if} \quad i = i'. \tag{1-2}$$

Let $a \in \mathcal{H}$ be a cyclic vector for V such that the spectral measure of a is σ_V. Take a unit vector $b \in L^2(G/H, \lambda_{G/H})$. Then for each $h \in H$,

$$\langle U_h(b \otimes a), b \otimes a\rangle = \langle V_ha, a\rangle$$

This implies that π projects the spectral measure of $b \otimes a$ into σ_V. This yields $\sigma'_{i,j} \circ \pi^{-1} = \sigma_{i,j}$ for each pair (i,j). Therefore

$$\sigma'_{i,j} \perp \sigma'_{i',j'} \quad \text{if} \quad i \neq i'. \tag{1-3}$$

Since $L^2(G/H, \lambda_{G/H}) \otimes \mathcal{H} = \bigoplus_{i \in M} \bigoplus_{j=1}^{i} L^2(G/H, \lambda_{G/H}) \otimes \mathcal{H}_{i,j}$, we deduce both (i) and (ii) from (1-2) plus (1-3). $\square$

Corollary 1.2. *If a G-action T is induced by an H-action S then $\mathcal{M}(T) = \mathcal{M}(S) \cup \{1\}$.*

The "extra" value 1 appears because $L^2_0(G/H, \lambda_{G/H}) \otimes 1$ is a U_T-cyclic subspace of $L^2_0(G/H \times X, \lambda_{G/H} \times \mu)$.

In the rest of this section we describe the self-joinings of induced actions and deduce some natural corollaries of this description. These results, which seem to be of independent interest, will not be used further in the paper.

Denote by V the action of G on the homogeneous space G/H by translations. The following proposition about induced actions was shown by R. Zimmer in [Zi].

Proposition 1.3.

(i) *Let T be an action of G on $(X, \mathfrak{B}, \mu)$. Then the action of G induced by $T \restriction H$ is isomorphic to the Cartesian product $V \times T$.*

(ii) *An action T of G is induced by an action of H if and only if T has a factor isomorphic to V.*

Recall that given a dynamical system (Z, ν, T), a $T \times T$-invariant measure ρ on the product space $Z \times Z$ such that the coordinate marginals of ρ are both equal to ν is called a *(2-fold) self-joining of T*. For the theory of joinings and notions like relative weak mixing, relative compactness, simplicity and centralizer we refer the reader to [JuR] and [KaT].

In the following corollary we describe the structure of self-joinings of induced actions.

Proposition 1.4. *Let T be a G-action induced by an ergodic H-action S (see* (1-1) *for the notation). Let ρ be an ergodic self-joining of T. Then $(Y \times X \times Y \times X, \rho, T \times T)$ is an induced G-action. More precisely, there are $\kappa \in J_2^e(S)$ and $g \in G$ such that*

$$\rho = \int_{G/H} \kappa \circ (S_{h(s(y),y)} \times S_{h(s(y),gy)}) \times \delta_y \times \delta_{gy}\, d\lambda_{G/H}(y).$$

ρ is the graph of an isomorphism if and only if so is κ. Hence two induced G-actions are isomorphic if and only if the underlying H-actions are isomorphic.

Proof. We use the notation from (1-1). The projection map $Y \times X \to Y$ intertwines T with V (see (1-1)). Therefore the projection ρ^* of ρ to $Y \times Y$ is an ergodic self-joining of V. Hence there is $g \in G$ such that $\rho^*(A \times B) = \lambda_{G/H}(A \times gB)$ for all measurable subsets $A, B \subset G/H$. Disintegrate now ρ with respect to ρ^*:

$$\rho = \int_{G/H} \kappa_y \times \delta_y \times \delta_{gy}\, d\lambda_{G/H}(y), \tag{1-4}$$

where $Y \times Y \ni (y, y) \mapsto \kappa_y$ is a measurable field of probability measures on $X \times X$ such that

$$\int_Y \kappa_y^{(1)} \times \delta_y \, d\lambda_{G/H}(y) = \int_Y \kappa_y^{(2)} \times \delta_{gy} \, d\lambda_{G/H}(y) = \mu \times \lambda_{G/H}, \tag{1-5}$$

where $\kappa_y^{(i)}$ is the i-th coordinate projection of κ_y for $i = 1, 2$ and every $y \in Y$. Since ρ is $T \times T$-invariant, we deduce from (1-4) and (1-1) that

$$\kappa_{g'y} = \kappa_y \circ (S_{h(g',y)} \times S_{h(g',gy)}) \tag{1-6}$$

for each $g' \in G$ at a.e. $y \in G/H$. Substituting $g' \in H$ into (1-6) we obtain that κ_y is invariant under $S \times S$ for a.a. $y \in G/H$. Since μ is ergodic under S, we deduce from (1-5) that $\kappa_y^{(1)} = \kappa_y^{(2)} = \mu$ for a.a. $y \in G/H$. Thus κ_y is a self-joining of S for a.a. $y \in G/H$. Since G acts transitively on Y, the equation (1-6) can be "resolved" in a standard way:

$$\kappa_y = \kappa \circ (S_{h(s(y),y)} \times S_{h(s(y),gy)}), \quad y \in G/H,$$

for certain self-joining κ of S (formally, put $\kappa = \kappa_H$ and $g' = s(y)$ into (1-6)). Moreover, κ is ergodic.

The remaining assertions of Proposition 1.4 follow immediately. $\square$

Corollary 1.5.

(i) *If S is either relatively weakly mixing or relatively compact with respect to some factor $\mathfrak{A}$ then T is either relatively weakly mixing or relatively compact (respectively) with respect to the factor induced by $\mathfrak{A}$.*

(ii) *T is simple if and only if S has pure point spectrum.*

(iii) $C(T) = \{(\mathrm{Id} \times R)T_g \mid g \in G, R \in C(S)\}$.

(iv) *If $\mathfrak{F}$ is a factor of T that contains the standard factor V then $\mathfrak{F}$ is an induced action of a factor of S.*

We note that T may also have factors which do not contain V (for instance in the case considered in Proposition 1.3(i)).

2. Preliminaries

We start with an important algebraic lemma. Let G be a countable Abelian group, H a subgroup of G and $v : G \to G$ a group automorphism. We set

$$L(G, H, v) := \{\#(\{v^i(h) \mid i \in \mathbb{Z}\} \cap H), h \in H \setminus \{0\}\}.$$

Algebraic Lemma 2.1 ([KwL], [Da3]). *Given any subset $E \subset \mathbb{N}$, there exist a countable Abelian group G, a subgroup $H \subset G$ and an automorphism $v : G \to G$ such that*

(i) *$E = L(G, H, v)$ and*

(ii) *the subgroup of $\widehat{v}$-periodic points in $\widehat{G}$ is countable and dense.*

We now recall the definition of rank one. Let $S = (S_g)_{g \in \mathbb{R}^d}$ be a measure preserving action of $\mathbb{R}^d$ on a standard σ-finite measure space $(Y, \mathfrak{C}, \nu)$.

Definition 2.2.

(i) A *Rokhlin tower or column* for S is a triple (A, f, F), where $A \in \mathfrak{C}$, F is a cube in $\mathbb{R}^d$ and $f : A \to F$ is a measurable mapping such that for any Borel subset $H \subset F$ and an element $g \in \mathbb{R}^d$ with $g + H \subset F$, one has $f^{-1}(g + H) = S_g f^{-1}(H)$.

(ii) S is said to be of *rank one (by cubes)* if there exists a sequence of Rokhlin towers (A_n, f_n, F_n) such that the volume of F_n goes to infinity and for any subset $C \in \mathfrak{C}$ of finite measure, there is a sequence of Borel subsets $H_n \subset F_n$ such that

$$\lim_{n\to\infty} \nu(C \triangle f_n^{-1}(H_n)) = 0.$$

Denote by $\mathcal{R} \subset X \times X$ the T-orbit equivalence relation. A Borel map α from $\mathcal{R}$ to a compact Abelian group K is called a *cocycle* of $\mathcal{R}$ if

$$\alpha(x, y) + \alpha(y, z) = \alpha(x, z) \quad \text{for all } (x, y), (y, z) \in \mathcal{R}.$$

Two cocycles $\alpha, \beta : \mathcal{R} \to K$ are *cohomologous* if there is a μ-conull subset $B \subset X$ such that

$$\alpha(x, y) = \phi(x) + \beta(x, y) - \phi(y) \qquad \text{for all } (x, y) \in \mathcal{R} \cap (B \times B).$$

for a Borel map $\phi : X \to K$. Given a cocycle $\alpha : \mathcal{R} \to K$ and a closed subgroup $H \subset K$, we can define a new flow $T^{\alpha,H} = (T_t^{\alpha,H})_{t\in\mathbb{R}}$ on the space $(X \times K/H, \mu \times \lambda_{K/H})$ by setting

$$T_t^{\alpha,H}(x, k + H) = (T_t x, \alpha(T_t x, x) + k + H).$$

This flow is called *a compact group extension* of T. Given a character $\chi \in \widehat{K}$, we denote by $U_{T^\alpha,\chi}$ the following unitary representation of $\mathbb{R}$ in $L^2(X, \mu)$:

$$(U_{T^\alpha,\chi}(t)f)(x) := \chi(\alpha(T_{-t}x, x))f(T_{-t}x).$$

There is a natural decomposition of $U_{T^{\alpha,H}}$ into an orthogonal sum

$$U_{T^{\alpha,H}} = \bigoplus_{\chi\in\widehat{K/H}} U_{T^\alpha,\chi},$$

where $\widehat{K/H}$ is considered as a subgroup of $\widehat{K}$.

If a transformation S commutes with T (i.e. $S \in C(T)$) then a cocycle $\alpha \circ S : \mathcal{R} \to K$ is well defined by $\alpha \circ S(x, y) := \alpha(Sx, Sy)$. The important cohomology equation on α mentioned in Section 0 can now be stated as follows

$$\text{(2-1)} \qquad \alpha \circ S \ \text{ is cohomologous to } \ v \circ \alpha$$

for some $S \in C(T)$ and a group automorphism $v : K \to K$.

3. (C, F)-flows and (C, F)-cocycles and their rank-one counterparts

To prove Main Theorem we will use the (C, F)-construction (see [Da1] and references therein). We now briefly outline its formalism. Then we will interpret it in the standard language of cutting-and-stacking. Let two sequences $(C_n)_{n>0}$ and $(F_n)_{n\geq 0}$ of subsets in $\mathbb{R}$ be given such that:

— $F_n = [0, h_n)$, $h_0 = 1$,
— C_n is finite, $\#C_n > 1$, $\min C_n = 0$,
— $F_n + C_{n+1} \subset [0, h_{n+1} - 1)$,
— $(F_n + c) \cap (F_n + c') = \emptyset$ if $c \neq c'$, $c, c' \in C_{n+1}$,
— $\lim_{n\to\infty} \frac{h_n}{\#C_1 \cdots \#C_n} < \infty$.

Let $X_n := F_n \times C_{n+1} \times C_{n+2} \times \cdots$. Endow this set with the standard product Borel structure. The following map

$$(f_n, c_{n+1}, c_{n+2}) \mapsto (f_n + c_{n+1}, c_{n+2}, \dots)$$

is a Borel embedding of X_n into X_{n+1}. We now set $X := \bigcup_{n\geq 0} X_n$ and endow it with the inductive limit standard Borel structure. Given a Borel subset $A \subset F_n$, we denote by $[A]_n$ the following cylinder: $\{x = (f, c_{n+1}, \dots,) \in X_n \mid f \in A\}$. The family of all cylinders generates the entire σ-algebra $\mathfrak{B}$ on X.

Let $\mathcal{R}$ stand for the *tail* equivalence relation on X: two points $x, x' \in X$ are $\mathcal{R}$-equivalent if there is $n > 0$ such that $x = (f_n, c_{n+1}, \dots)$, $x' = (f'_n, c'_{n+1}, \dots) \in X_n$ and $c_m = c'_m$ for all $m > n$. Of course, $\mathcal{R}$ is a Borel subset of $X \times X$. It is easy to see that there is only one probability (non-atomic) Borel measure μ on X which is invariant under $\mathcal{R}$. This means that every Borel isomorphism of X whose graph is a subset of $\mathcal{R}$ preserves μ. We note that the restriction of μ on X_n is an infinite product $\nu_n \times \kappa_{n+1} \times \kappa_{n+2} \times \cdots$, where κ_i is the equidistribution on C_i and ν_{n+1} is a measure proportional to $\lambda_{\mathbb{R}} \restriction F_n$. Hence for each $n \geq 0$ and a subset $A \subset F_n$,

$$\mu([A]_n)/\mu(X_n) = \lambda_{\mathbb{R}}(A)/h_n.$$

We now isolate a subset $\widetilde{X} \subset X$ such that

$$\widetilde{X} \cap X_n := \{x = (f_n, c_{n+1}, c_{n+2}, \dots) \in X_n \mid c_k \neq 0 \text{ infinitely often}\}.$$

Then X_n is Borel, $\mathcal{R}$-saturated and $\mu(\widetilde{X}) = 1$. Now we define a Borel flow $T = (T_t)_{t\in\mathbb{R}}$ on $\widetilde{X}$ by setting

$$T_t(f_n, c_{n+1}, \dots) := (t + f_n, c_{n+1}, \dots) \text{ whenever } t + f_n < h_n,\ n > 0.$$

This formula defines T_t partly on $\widetilde{X}$. When $n \to \infty$, T_t extends to the entire $\widetilde{X}$. It is easy to see that the mapping $\widetilde{X} \times \mathbb{R} \ni (x, t) \mapsto T_t x \in \widetilde{X}$ is Borel and $T_{t_1} T_{t_2} = T_{t_1+t_2}$ for all $t_1, t_2 \in \mathbb{R}$. Moreover, the T-orbit equivalence relation coincides with $\mathcal{R} \restriction \widetilde{X}$. It follows that T is μ-preserving. In what follows we do not distinguish objects (sets, transformations, etc.) if they agree a.e. That is why we consider that T is defined on the entire X.

Definition 3.1. We call T *the* (C, F)*-flow* associated with $(C_{n+1}, F_n)_{n\geq 0}$.

It is easy to see that T is of rank one (see the comment after Lemma 3.3). Hence it is free and ergodic.

We recall a concept of (C,F)-cocycle (see [Da2]). Given a sequence of maps $\alpha_n : C_n \to K$, $n = 1, 2, \dots$, we first define a Borel cocycle $\alpha : \mathcal{R} \cap (X_0 \times X_0) \to K$ by setting

$$\alpha(x, x') := \sum_{n>0} (\alpha_n(c_n) - \alpha_n(c'_n)),$$

whenever $x = (0, c_1, c_2, \dots) \in X_0$, $x' = (0, c'_1, c'_2, \dots) \in X_0$ and $(x, x') \in \mathcal{R}$. To extend α to the entire $\mathcal{R}$, we first define a map $\pi : X \to X_0$ as follows. Given $x \in X$, let n be the least positive integer such that $x \in X_n$. Then $x = (f_n, c_{n+1}, \dots) \in X_n$. We set

$$\pi(x) := (\underbrace{0, \dots, 0}_{n+1 \text{ times}}, c_{n+1}, c_{n+2}, \dots) \in X_0.$$

Of course, $(x, \pi(x)) \in \mathcal{R}$. Now for each pair $(x, y) \in \mathcal{R}$, we let

$$\alpha(x, y) := \alpha(\pi(x), \pi(y)).$$

It is easy to verify that α is a well defined cocycle of $\mathcal{R}$ with values in K.

Definition 3.2. We call α *the (C,F)-cocycle associated with* $(\alpha_n)_{n=1}^{\infty}$.

Suppose we have an invertible measure preserving transformation S of (X, μ) such that S maps bijectively $\mathcal{R}(x)$ on $\mathcal{R}(Sx)$ for μ-a.a. $x \in X$. (This condition holds, for instance, if $S \in C(T)$.) Then for each cocycle $\alpha : \mathcal{R} \to K$, we can define a cocycle $\alpha \circ S$ by setting

$$\alpha \circ S(x, y) := \alpha(Sx, Sy), \quad (x, y) \in \mathcal{R}.$$

Adapting the argument from [Da2, Section 4] we obtain the following lemma.

Lemma 3.3. *Let $\bar{z} = (z_n)_{n+1}^{\infty}$ be a sequence of positive reals. Suppose that*

$$\sum_{n>0} \#(C_n \triangle (C_n - z_n))/\#C_n < \infty.$$

For each $m > 0$, we set

$$X_m^{\bar{z}} := [0, h_m - z_1 - \dots - z_m) \times \prod_{n>m} (C_n \cap (C_n - z_n)) \subset X_m.$$

Then the transformation $S_{\bar{z}}$ of (X, μ) is well defined by setting

$$S_{\bar{z}}(x) := (z_1 + \dots + z_m + f_m, z_{m+1} + c_{m+1}, z_{m+2} + c_{m+2}, \dots) \tag{3-1}$$

for all $x = (f_m, c_{m+1}, c_{m+2}, \dots) \in X_m^{\bar{z}}$, $m = 1, 2, \dots$. Moreover, $S_{\bar{z}}$ commutes with T and $T^{z_1 + \dots + z_m} \to S_{\bar{z}}$ weakly as $m \to \infty$.

Let v be a continuous group automorphism of K and let

$$C_m^{\circ} := \{c \in C_m \cap (C_m - z_m) \mid \alpha_m(c + z_m) = v(\alpha_m(c))\}.$$

If

$$\sum_{n>0} (1 - \#C_n^{\circ}/\#C_n) < \infty \tag{3-2}$$

then the cocycle $\alpha \circ S_{\bar{z}}$ is cohomologous to $v \circ \alpha$.

We now interpret the (C, F)-construction in terms of the common geometrical cutting-and-stacking techniques. Recall that we have two sequences $(F_n)_{n\geq 0}$ and $(C_n)_{n>0}$ as above. Our purpose is to construct inductively a sequence of towers X_n, $n = 1, 2, \dots$. On the $(n-1)$-st step of the construction we have a tower X_{n-1}, which is a rectangular of height h_n. Denote the width of X_{n-1} by w_{n-1}. Let $C_n = \{c_{n,1}, c_{n,2}, \dots, c_{n,r_n}\}$ with $0 = c_{n,1} < \cdots < c_{n,r_n}$. We cut X_{n-1} into r_n subtowers of equal width $w_n := w_{n-1}/r_n$ and call them *copies* of X_{n-1}. Enumerate these copies from the left to the right. Then we put a rectangle of height $c_{n,i+1} - c_{n,i} - h_{n-1}$ and width w_n over the i-th copy of X_{n-1}, $1 \leq i < r_n$, and a rectangle of height $h_n - c_{n,r_n} - h_{n-1}$ and width w_n over the r_n-th copy of X_{n-1}. These additional rectangles are called "spacers". Thus we obtain a family of enumerated r_n towers of the same width but of (possibly) different length which is no less than h_{n-1}. Stack them in the following way: put the second tower over the top of the first tower, the third tower on the top of the second one and so on. Thus we obtain a new tower of height h_n and width w_n and call it the n-th tower.

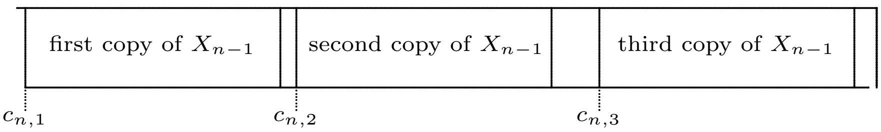

FIGURE 3.1. n-th tower X_n.

We draw an example of tower X_n (turned horizontally) with $r_n = 3$ at Figure 3.1.

Continuing this process infinitely many times we construct $X_1 \subset X_2 \subset \cdots$. Since X_n is embedded into $\mathbb{R}^2$, we endow it with the induced Lebesgue measure, say μ_n. Let (X, μ) be the inductive limit of the sequence $(X_1, \mu_1) \subset (X_2, \mu_2) \subset \cdots$. Now we define a flow $T = (T_t)_{t\in\mathbb{R}}$ on X geometrically as follows: T_t moves a point $x \in X_n$ up with a unit speed until it reaches the top of X_n, $n = 1, 2, \dots$. Then T is well defined on X (more precisely, on a μ-conull subset of X).

To define a (C, F)-cocycle of T, suppose that a sequence of maps $\alpha_n : C_n \to K$ is given. Given $x \in X$ and $n > 0$, we set $c_n(x) := c_{n,i} \in C_n$ if x belongs to the i-th copy of X_{n-1} in X_n. If x does not belong to any copy of X_n then we set $c_n(x) := 0$. Then two points $x, x' \in X$ are T-orbit equivalent if and only if $c_m(x) = c_m(x')$ for all sufficiently large m. We now put $\alpha(x, x') := \sum_{m>0}(\alpha_m(c_m(x)) - \alpha_m(c_m(x')))$. Then α is a cocycle of T, namely the (C, F)-cocycle associated with $(\alpha_n)_{n=1}^{\infty}$. The most important property of α is that if x belongs to the i-th copy of X_{n-1}, $i \neq r_n$, and $t = c_{n,i+1} - c_{n,i}$ then $\alpha(T_t x, x) = \alpha_n(c_{n,i+1}) - \alpha_n(c_{n,i})$.

We note that the (C, F)-cocycles for T are analogues of the Morse cocycles (see [Go] and references therein) called also rank-one cocycles or cocycles of product type for $\mathbb{Z}$-actions.

4. Realization of sets containing 1 as spectral multiplicities

Let E be a subset of positive integers. By Algebraic Lemma 2.1, there exist a compact Polish Abelian group K, a closed subgroup H of K and a continuous automorphism v of K such that

$$E = L(\widehat{K}, \widehat{K/H}, \widehat{v}).$$

The subgroup of v-periodic points in K will be denoted by $\mathcal{K}$. It is countable and dense in K by Lemma 2.1. Let ξ_1 and ξ_2 be two rationally independent positive reals in $\mathbb{R}$. Fix a partition

$$\mathbb{N} = \mathcal{W}_1 \sqcup \mathcal{W}_2 \sqcup \bigsqcup_{a\in\mathcal{K}} \mathcal{N}_a$$

of $\mathbb{N}$ into infinite subsets. To construct the desired realization we define inductively a sequence $(C_n, h_n, \alpha_n)_{n=1}^{\infty}$, where C_n is a finite subset of $\mathbb{R}$, h_n is a positive real and $\alpha_n : C_n \to \mathbb{R}$ is a mapping. Suppose we have already constructed this sequence up to index n. Consider two cases.

If $n+1 \in \mathcal{N}_a$ for some $a \in \mathcal{K}$ then we denote by m_a the least positive period of a under v. Now we set

$$\begin{gathered} z_{n+1} := m_a n h_n, \quad r_n := n^3 m_a, \\ C_{n+1} := h_n \cdot \{0, 1, \dots, r_n - 1\}, \\ h_{n+1} := r_n h_n + 1, \end{gathered}$$

Let $\alpha_{n+1} : C_{n+1} \to K$ be any map satisfying the following conditions

(A1) $\alpha_{n+1}(c + z_{n+1}) = v \circ \alpha_{n+1}(c)$ for all $c \in C_{n+1} \cap (C_{n+1} - z_{n+1})$,
(A2) for each $0 \le i < m_a$, there is a subset $C_{n+1,i} \subset C_{n+1}$ such that

$$\begin{gathered} C_{n+1,i} - h_n \subset C_{n+1}, \\ \alpha_{n+1}(c) = \alpha_{n+1}(c - h_n) + v^i(a) \text{ for all } c \in C_{n+1,i} \text{ and} \\ \left| \frac{\#C_{n+1,i}}{\#C_{n+1}} - \frac{1}{m_a} \right| < \frac{2}{n m_a}. \end{gathered}$$

If $n+1 \in \mathcal{W}_i$ for $i = 1, 2$ then we set

$$\begin{gathered} C_{n+1} := \{j h_n \mid 0 \le j < n\} \sqcup \{j(h_n + \xi_i) + n h_n \mid 0 \le j < n\}, \\ h_{n+1} := 2n h_n + n\xi_i, \\ \alpha_{n+1}(c) := 1_K \quad \text{for all } c \in C_{n+1}. \end{gathered}$$

Thus, $C_{n+1}, h_{n+1}, \alpha_{n+1}$ are completely defined.

Denote by (X, μ, T) the (C, F)-flow associated with the sequence $(C_{n+1}, F_n)_{n\ge 0}$, where $F_n := [0, h_n)$. Let $\mathcal{R}$ stand for the tail equivalence relation (or, equivalently, the T-orbit equivalence relation) on X. Denote by $\alpha : \mathcal{R} \to K$ the cocycle of $\mathcal{R}$ associated with the sequence $(\alpha_n)_{n>0}$. Let $\lambda_{K/H}$ stand for the Haar measure on K/H. We denote by $T^{\alpha,H}$ the following flow on the space $(X \times K/H, \lambda_{K/H})$:

$$T_t^{\alpha,H}(x, k + H) := (T_t x, \alpha(T_t x, x) + k + H), \quad t \in \mathbb{R}.$$

Our purpose in this section is to prove the following theorem.

Theorem 4.1. $\mathcal{M}(T^{\alpha,H}) = E \cup \{1\}$.

Since

$$\sum_{n>0} \frac{\#(C_n \triangle (C_n - z_n))}{\# C_n} = \sum_{n>0} \frac{2}{n^2},$$

it follows from Lemma 3.3 that the transformation $S_{\bar{z}}$ of (X, μ) is well defined by the formula (3-1) and $S_{\bar{z}} \in C(T)$.

It follows from (A1) and the definition of C_{n+1} that (3-2) is satisfied. Hence by Lemma 3.3,

$$\text{the cocycle } \alpha \circ S_{\bar{z}} \text{ is cohomologous to } v \circ \alpha. \tag{4-1}$$

We need more notation. Given $a \in \mathcal{K}$ and $\chi \in \widehat{K}$, let

$$l_\chi(a) := m_a^{-1} \sum_{i=0}^{m_a - 1} \chi(v^i(a)),$$

where m_a stands for the least positive period of a under v.

Lemma 4.2. *Let $\chi \in \widehat{K}$. Then*

(i) *$U_{T^\alpha,\chi}(h_n) \to l_\chi(a) \cdot I$ as $\mathcal{N}_a - 1 \ni n \to \infty$, $a \in \mathcal{K}$.*
(ii) *$U_{T^\alpha,\chi}(h_n) \to 0.5(I + U_{T^\alpha,\chi}(-\xi_j))$ as $\mathcal{W}_j - 1 \ni n \to \infty$, $j = 1, 2$.*

Proof. We will show only (i). The other claim is shown in a similar way. Let $n \in \mathcal{N}_a$. Take any subset $A \subset F_n$. We note that $[A]_n = \bigsqcup_{c \in C_{n+1}} [A + c]_{n+1}$. Geometrically, the set $[A + c]_{n+1}$ is the intersection of the cylinder $[A]_n$ with the corresponding (to c) copy of the "tower" X_n in X_{n+1}. For each $0 \le i < m_a$, consider the set $\bigsqcup_{c \in C_{n+1,i}} [A + c]_{n+1} = [A + C_{n+1,i}]$. These sets are mutually disjoint and their union is "almost" the entire $[A]_n$ in view of the inequality from (A2). Moreover, by (A2), if $x \in [A + C_{n+1,i}]_{n+1,i}$ then $\alpha(x, T_{-h_n} x) = v^i(a)$. Therefore

$$\begin{aligned} U_{T^\alpha,\chi}(h_n) 1_{[A]_n}(x) &= \sum_{i=0}^{m_a-1} \chi(\alpha(x, T_{-h_n} x)) 1_{[A + C_{n+1,i}]_{n+1}}(T_{-h_n} x) + \bar{o}(x) \\ &= \sum_{i=0}^{m_a-1} \chi(v^i(a)) 1_{[A + C_{n+1,i} + h_n]_{n+1}}(x) + \bar{o}(x) \end{aligned}$$

where $\bar{o}(x)$ is a function whose L^2-norm is small. We note that $1_{[A + C_{n+1,i} + h_n]_{n+1}} = 1_{[F_n + C_{n+1,i} + h_n]_{n+1}} 1_{[A]_n}$ Hence

$$U_{T^\alpha,\chi}(h_n) - \sum_{i=0}^{m_a-1} \chi(v^i(a)) 1_{[F_n + C_{n+1,i} + h_n]_{n+1}} \to 0$$

weakly as $\mathcal{N}_a - 1 \ni n \to \infty$, where the function $1_{[F_n + C^a_{n+1,i} + h_n]_{n+1}} \in L^\infty(X, \mu)$ is considered as a multiplication operator in $L^2(X, \mu)$.

It remains to use the inequalities from (A2) and a standard fact that for any sequence $C'_n \subset C_n$ such that $\# C'_n / \# C_n \to \delta$ for some $\delta > 0$ we have

$$1_{[F_n + C'_n]_n} \to \delta I \quad \text{weakly as } n \to \infty.$$

$\square$

Proof of Theorem 4.1. We first verify that T^α is weakly mixing. Let $U_{T^\alpha}(t)f = \exp(i\lambda t)f$ for some $f \in L^2(X \times K)$, $f \neq 0$ and $\lambda \in \mathbb{R}$. It follows from Lemma 4.2(ii) that

$$U_{T^\alpha}(h_n) \to 0.5(I + U_{T^\alpha}(-\xi_j))$$

and hence

$$\exp(ih_n\lambda) \to 0.5(1 + \exp(-i\lambda\xi_j))$$

as $\mathcal{W}_j - 1 \ni n \to \infty$, $j = 1, 2$. Therefore $|1 + \exp(-i\lambda\xi_j)| = 2$ which implies $\exp(-i\lambda\xi_j) = 1$ for $j = 1, 2$. Since ξ_1 and ξ_2 are rationally independent, $\lambda = 0$. It remains to show that T^α is ergodic. If $\chi \neq 1$ then there is $a \in \mathcal{K}$ with $l_\chi(a) \neq 1$. If $f \in L^2(X, \mu)$ is invariant under $U_{T^\alpha,\chi}$ then Lemma 4.2(i) yields $f = l_\chi(a)f$. Hence $f = 0$. If $\chi = 1$ then $U_{T^\alpha,\chi} = U_T$. Since T is ergodic, each $U_{T^\alpha,\chi}$-invariant function is constant. Thus, we have shown that U_{T^α} is weakly mixing. Hence $U_{T^{\alpha,H}}$ is also weakly mixing.

To show that $\mathcal{M}(T^{\alpha,H}) = E \cup \{1\}$ we consider a natural decomposition of $U_{T^{\alpha,H}}$ into an orthogonal sum

$$U_{T^{\alpha,H}} = \bigoplus_{\chi \in \widehat{K/H}} U_{T^\alpha,\chi}.$$

It is enough to prove the following:

(a) $U_{T^\alpha,\chi}$ has a simple spectrum for each χ,
(b) $U_{T^\alpha,\chi}$ and $U_{T^\alpha,\xi}$ are unitarily equivalent if χ and ξ belong to the same $\widehat{v}$-orbit,
(c) the maximal spectral types of $U_{T,\chi}$ and $U_{T,\xi}$ are mutually singular if χ and ξ belong to different $\widehat{v}$-orbits.

For each $\epsilon > 0$ and $n > 0$, there are a partition of F_n into intervals $\Delta_0, \dots, \Delta_{M_n}$ and reals $t_1, \dots, t_{M_n}$ such that $\max_j \operatorname{diam} \Delta_j < \epsilon$, $\Delta_j = T_{t_j}\Delta_0$ and the mapping $[\Delta_j]_n \ni x \mapsto \alpha(T_{-t_j}x, x) \in K$ is constant for each $1 \leq j \leq M_n$. This implies (a).

It is straightforward that (4-1) implies (b).

If χ and η are non-equivalent then there is $a \in \mathcal{K}$ such that $l_\chi(a) \neq l_\eta(a)$. Moreover, $U_{T,\chi}^{h_n} \to l_\chi(a)I$ and $U_{T,\eta}^{h_n} \to l_\eta(a)I$ as $\mathcal{N}_a - 1 \ni n \to \infty$ by Lemma 4.2(i). Hence the maximal spectral types of $U_{T,\eta}$ and $U_{T,\chi}$ are mutually singular. Thus (c) holds. $\square$

Now we are going to show the following claim.

Proposition 4.3. *$\mathcal{M}(T_t) = E$ for each $t \neq 0$.*

For that we need an auxiliary statement from [LeP]. Given a Borel measure σ on $\mathbb{R}$, we let $A_\sigma := \{t \in \mathbb{R} \mid \sigma * \delta_t \not\perp \sigma\}$.

Lemma 4.4([LeP]). *Let σ be a finite Borel measure on $\mathbb{R}$. If there are an analytic function a on $\mathbb{R}$ and a sequence of continuous characters $\xi_n \in \widehat{\mathbb{R}}$ such that $\xi_n \to \infty$ in $\widehat{\mathbb{R}}$ and $\xi_n \to a$ weakly in $L^2(\mathbb{R}, \sigma)$ then for each $t_0 \in A_\sigma$ there exists $c \in \mathbb{C}$ with $|c| = 1$ and $a(t + t_0) = ca(t)$ for each $t \in \mathbb{R}$.*

Proof. [1] Given two measures λ and κ on $\mathbb{R}$, we write $\lambda \ll \kappa$ if λ is absolutely continuous with respect to κ and the Radon-Nikodym derivative $d\lambda/d\kappa$ is bounded.

[1]This is a fragment of the proof of Proposition 5 from [LeP]. We include it here by a recommendation of the referee.

Fix $t_0 \in A(\sigma)$. Passing to a subsequence if necessary, we can assume that $\langle \xi_n, t_0 \rangle \to c$ for some $c \in \mathbb{C}$, $|c| = 1$. Take an arbitrary positive finite Borel measure λ so that $\lambda \ll \sigma$ and $\lambda * \delta_{-t_0} \ll \sigma$. Since $\xi_n \to a$ weakly in $L^2(\mathbb{R}, \sigma)$, it follows that $\xi_n \to a$ weakly in $L^2(\mathbb{R}, \kappa)$ for each measure $\kappa \ll \sigma$. Therefore

$$\begin{aligned}\int \langle \xi_n, t + t_0 \rangle \, d\lambda(t) &= \int \langle \xi_n, t \rangle \, d(\lambda * \delta_{-t_0})(t) \\ &\to \int a(t) \, d(\lambda * \delta_{-t_0})(t) \\ &= \int a(t + t_0) \, d\lambda(t).\end{aligned}$$

On the other hand,

$$\int \langle \xi_n, t + t_0 \rangle \, d\lambda(t) \to c \int a(t) \, d\lambda(t).$$

Since the above convergences also take place for each $\lambda_1 \ll \lambda$, we obtain $a(t+t_0) = ca(t)$ for λ-a.e. $t \in \mathbb{R}$. Since λ is continuous and a is analytic, $a(t + t_0) = ca(t)$ for all $t \in \mathbb{R}$. $\square$

Proof of Proposition 4.3. Denote by σ_T a probability measure of maximal spectral type for T. We first show that $A_{\sigma_T} = \{0\}$. It follows from Lemma 4.2(ii) that

$$U_{T^{\alpha,H}}(h_n) \to 0.5(I + U_{T^{\alpha,H}}(-\xi_j))$$

weakly as $\mathcal{W}_j - 1 \ni n \to \infty$, $j = 1, 2$. We deduce from this and Lemma 4.4 that for each $t_0 \in A_{\sigma_T}$ and $j = 1, 2$, there exists a complex number c_j such that

$$1 + \exp(2\pi i \xi_j (t + t_0)) = c_j (1 + \exp(2\pi i \xi_j t))$$

for all $t \in \mathbb{R}$. This yields $c_j = 1$ and $\exp(2\pi i \xi_j t_0) = 1$ for $j = 1, 2$. Since ξ_1 and ξ_2 are rationally independent, $t_0 = 0$.

Thus if $0 \neq t \in \mathbb{R}$ then $\sigma_T * \delta_t \perp \sigma_T$. Hence the natural projection $\mathbb{R} \to \mathbb{R}/t\mathbb{Z}$ is one-to-one on a subset of full σ_T-measure. This implies that $\mathcal{M}(T^{\alpha,H}) = \mathcal{M}(T^{\alpha,H}_t)$. $\square$

5. Realization of sets containing 2 as spectral multiplicities

Now let E be a subset of $\mathbb{N}$ such that $1 \notin E$. In this section we will realize the set $E \cup \{2\}$. To this end we combine the technique of compact group extensions developed in the previous section with a technique of "Cartesian products". We illustrate the latter with a couple of auxiliary lemmata.

Lemma 5.1. *Let T be a weakly mixing flow with a simple spectrum. Let ξ_1, ξ_2 be two rationally independent reals. Suppose that the weak closure $WC(U_T)$ of the group $\{U_T(t) \mid t \in \mathbb{R}\}$ contains the following operators:*

$$0.5(I + U_T(j\xi_1)), \quad 0.5(I + U_T(\xi_2)) \quad \text{and} \quad 0.5(I + U_T(\xi_2 - \xi_1)), \tag{5-1}$$

$j = 1, 2$. Then the product flow $T \times T := (T_t \times T_t)_{t \in \mathbb{R}}$ has a homogeneous spectrum of multiplicity 2 in the orthocomplement to the constants.

Proof. Let h be a cyclic vector for U_T. Denote by $\mathcal{C}$ the closure of the span of 3 vectors $h\otimes h, U_T(\xi_1)h\otimes h$ and $U_T(\xi_2)h\otimes h$. It follows from (5-1) that $\mathcal{C}$ is invariant under the following operators:

$$U_T(j\xi_1)\otimes I + I\otimes U_T(j\xi_1), \quad j=1,2, \tag{5-2}$$

$$U_T(\xi_2)\otimes I + I\otimes U_T(\xi_2), \tag{5-3}$$

$$U_T(\xi_2-\xi_1)\otimes I + I\otimes U_T(\xi_2-\xi_1). \tag{5-4}$$

Slightly modifying the argument from [Ag] and [Ry1], we deduce from (5-2) and (5-3) that

$$U_T(n\xi_1)h\otimes h\in\mathcal{C} \quad \text{and} \quad U_T(n\xi_2)h\otimes h\in\mathcal{C}$$

for all $n\in\mathbb{Z}$. Applying (5-4) to $U(\xi_1)h\otimes h$ we obtain that $U_T(2\xi_1-\xi_2)h\otimes h\in\mathcal{C}$. Applying (5-2) with $j=-2$ step by step infinitely many times and then with $j=2$ infinitely many times we obtain that $U(2n\xi_1-\xi_2)h\otimes h\in\mathcal{C}$ for each $n\in\mathbb{Z}$. Next, applying (5-3) to $U(2\xi_1-\xi_2)h\otimes h$, we deduce that $U(2\xi_1-2\xi_2)\in\mathcal{C}$. Then again apply infinitely many times (5-2) with $j=-2$ and $j=2$ to obtain $U(2n\xi_1-2\xi_2)h\otimes h\in\mathcal{C}$. And so on. Finally, we obtain that

$$U(2n\xi_1+m\xi_2)h\otimes h\in\mathcal{C} \quad \text{for all } n,m\in\mathbb{Z}.$$

Hence $U(t)h\otimes h\in\mathcal{C}$ for all $t\in\mathbb{R}$. Since h is cyclic for U, it follows that $\mathcal{H}\otimes h\subset\mathcal{C}$ and therefore $\mathcal{C}=\mathcal{H}\otimes\mathcal{H}$.

Denote by m the spectral multiplicity function for $(T_t\times T_t)_{t\in\mathbb{R}}$ and denote by σ the measure of maximal spectral type for T. By the above, $m(\lambda)\le 3$ for $\sigma*\sigma$-a.a. $\lambda\in\mathbb{R}$.

On the other hand, since T is weakly mixing and $\sigma\times\sigma=\int_{\mathbb{R}}\sigma_\lambda\,d\sigma*\sigma(\lambda)$ and σ_λ is invariant under the flip mapping $\mathbb{R}^2\ni(y,z)\mapsto(z,y)\in\mathbb{R}^2$, it follows that $m(\lambda)\in\{2,4,\dots\}\cup\{\infty\}$. Hence $m(\lambda)=2$ a.e. $\square$

Lemma 5.2. *Let U and V be unitary representations of $\mathbb{R}$ with simple spectrum. Assume that there are sequences $a_n\to\infty$, $b_n\to\infty$ $a_n'\to\infty$ and $b_n'\to\infty$ such that*

(i) *$U(a_n)\to 0.5(I+U(\xi))$ and $V(a_n)\to 0.5(I+V(\xi))$,*
(ii) *$U(b_n)\to 0.5(dI+U(\xi))$ and $V(b_n)\to 0.5(eI+V(\xi))$,*
(iii) *$U(a_n')\to 0.5(I+U(\eta))$ and $V(a_n')\to 0.5(I+V(\eta))$ and*
(iv) *$U(b_n')\to 0.5(d'I+U(\eta))$ and $V(b_n')\to 0.5(e'I+V(\eta))$*

for some $\xi,\eta,d,e,d',e'\in\mathbb{R}$. If $d\ne e$, $d'\ne e'$ and ξ and η are rationally independent then $U\otimes V$ has also a simple spectrum.

Proof. Let v_1 and v_2 be cyclic vectors for U and V. Denote by $\mathcal{C}$ the $U\otimes V$-cyclic subspace generated by $v_1\otimes v_2$. It follows from (i) and (ii) that

$$(I+U(\xi))v_1\otimes(I+V(\xi))v_2\in\mathcal{C}.$$
$$(dI+U(\xi))v_1\otimes(eI+V(\xi))v_2\in\mathcal{C}.$$

Hence $U(\xi)v_1\otimes v_2+v_1\otimes V(\xi)v_2\in\mathcal{C}$ and $dU(\xi)v_1\otimes v_2+ev_1\otimes V(\xi)v_2\in\mathcal{C}$. This implies, in particular that $U(\xi)v_1\otimes v_2\in\mathcal{C}$. In a similar way, $U(-\xi)v_1\otimes v_2\in\mathcal{C}$. Thus, $(U(\xi)\otimes I)\mathcal{C}=\mathcal{C}$. In a similar way, we deduce from (iii) and (iv) that $(U(\eta)\otimes I)\mathcal{C}=\mathcal{C}$. Hence $(U(n\xi+m\eta)\otimes I)\mathcal{C}=\mathcal{C}$ for all $n,m\in\mathbb{Z}$. Since η and ξ

are rationally independent, $\mathcal{C}$ is invariant under the unitary representation $U \otimes I$. Since $\mathcal{C} \ni v_1 \otimes v_2$, it follows that $\mathcal{C} \supset \mathcal{H}_1 \otimes v_2$. Since $\mathcal{C}$ is $U \otimes V$-invariant, we deduce that $\mathcal{C} \supset \mathcal{H}_1 \otimes \mathcal{H}_2$ and hence $\mathcal{C} = \mathcal{H}_1 \otimes \mathcal{H}_2$. $\square$

Let $K, H, v, \mathcal{K}, \xi_1, \xi_2$ be as in the previous section. We will assume that $\xi_2 > \xi_1$ and put $\xi_3 := \xi_2 - \xi_1$. Fix a partition

$$\mathbb{N} = \bigsqcup_{i=1}^{3} \bigsqcup_{a \in \mathcal{K}} \mathcal{M}_{a,i} \sqcup \mathcal{N}_a$$

of $\mathbb{N}$ into infinite subsets. As in the previous section, to construct the desired realization we define inductively a sequence $(C_n, h_n, \alpha_n)_{n=1}^{\infty}$, where C_n a finite subset of $\mathbb{R}$, h_n is a positive real and $\alpha_n : C_n \to \mathbb{R}$ a mapping. Suppose we have already constructed this sequence up to index n. Consider two cases.

Case 1. If $n + 1 \in \mathcal{N}_a$ for some $a \in \mathcal{K}$ then we denote by m_a the least positive period of a under v. Now we set

$$z_{n+1} := m_a n h_n, \quad r_n := n^3 m_a,$$
$$C_{n+1} := h_n \cdot \{0, 1, \dots, r_n - 1\},$$
$$h_{n+1} := r_n h_n,$$

Let $\alpha_{n+1} : C_{n+1} \to K$ be any map satisfying the following conditions

(A1) $\alpha_{n+1}(c + z_{n+1}) = v \circ \alpha_{n+1}(c)$ for all $c \in C_{n+1} \cap (C_{n+1} - z_{n+1})$,

(A2) for each $0 \le i < m_a$ there is a subset $C_{n+1,i} \subset C_{n+1}$ such that

$$C_{n+1,i} - h_n \subset C_{n+1},$$
$$\alpha_{n+1}(c) = \alpha_{n+1}(c - h_n) + v^i(a) \text{ for all } c \in C_{n+1,i} \text{ and}$$
$$\left| \frac{\#C_{n+1,i}}{\#C_{n+1}} - \frac{1}{m_a} \right| < \frac{2}{n m_a}.$$

Case 2. If $n + 1 \in \mathcal{M}_{a,i}$ for some $a \in \mathcal{K}$ and $i = 1, 2, 3$ then we denote by m_a the least positive period of a under v. Now we set

$$z_{n+1} := m_a n(2h_n + \xi_i),$$
$$D^1_{n+1} := h_n \cdot \{0, 1, \dots, m_a n - 1\},$$
$$D^2_{n+1} := \{j(h_n + \xi_i) + m_n n h_n \mid 0 \le j < m_a n\},$$
$$C_{n+1} := \bigsqcup_{j=0}^{n^2-1} (j z_{n+1} + (D^1_{n+1} \sqcup D^2_{n+1})),$$
$$h_{n+1} := m_a n^3 (2h_n + \xi_i),$$

Let $\alpha_{n+1} : C_{n+1} \to K$ be any map satisfying the following conditions

(B1) $\alpha_{n+1}(c + z_{n+1}) = v \circ \alpha_{n+1}(c)$ for each $c \in C^1_{n+1} \cap (C^1_{n+1} - z_{n+1})$,

(B2) for each $0 \le l < m_a$ there is a subset $D_{n+1,l} \subset D^1_{n+1}$ such that

$$D_{n+1,l} - h_n \subset D^1_{n+1},$$
$$\alpha_{n+1}(c) = \alpha_{n+1}(c - h_n) + v^l(a) \text{ for all } c \in D_{n+1,l} \text{ and}$$
$$\left| \frac{\#D_{n+1,l}}{\#D^1_{n+1}} - \frac{1}{m_a} \right| < \frac{2}{n m_a},$$

(B3) $\alpha_{n+1}(c) = 1_K$ for each $c \in D^2_{n+1}$.

Thus, $C_{n+1}, h_{n+1}, \alpha_{n+1}$ are completely defined.

Denote by (X, μ, T) the (C, F)-flow associated with the sequence $(C_{n+1}, F_n)_{n\geq 0}$, where $F_n := [0, h_n)$. Let $\mathcal{R}$ stand for the tail equivalence relation (or, equivalently, the T-orbit equivalence relation) on X. Denote by $\alpha : \mathcal{R} \to K$ the cocycle of $\mathcal{R}$ associated with the sequence $(\alpha_n)_{n>0}$. We denote by $T^{\alpha,H}$ the following flow on the space $(X \times K/H, \lambda_{K/H})$:

$$T_t^{\alpha,H}(x, k+H) := (T_t x, \alpha(T_t x, x) + k + H), \quad t \in \mathbb{R}.$$

The following lemma is an analogue of Lemma 4.2. It can be proved in a similar way by using (A2), (B2) and (B3). We leave details to the reader.

Lemma 5.3. *Let $a \in \mathcal{K}$. Then for each $\chi \in \widehat{K}$ and $j > 0$*

(i) $U_{T^\alpha,\chi}(h_n) \to l_\chi(a) \cdot I$ *as* $\mathcal{N}_a - 1 \ni n \to \infty$ *and*

(ii) $U_{T^\alpha,\chi}(jh_n) \to 0.5(l_\chi(ja)I + U_{T^\alpha,\chi}(-j\xi_i))$ *as* $\mathcal{M}_{a,i} - 1 \ni n \to \infty$.

Our purpose in this section is to prove the following theorem.

Theorem 5.4. *The transformation $T \times T^{\alpha,H}$ is weakly mixing and $\mathcal{M}(T \times T^{\alpha,H}) = E \cup \{2\}$.*

Proof. To show that $\mathcal{M}(T \times T^{\alpha,H}) = E \cup \{2\}$ we consider a natural decomposition of $U_{T\times T^{\alpha,H}}$ into an orthogonal sum

$$U_{T\times T^{\alpha,H}} = \bigoplus_{\chi\in\widehat{K/H}} (U_T \otimes U_{T,\chi}).$$

It is enough to prove the following:

(a) $U_T \otimes U_T$ has a homogeneous spectrum 2 in the orthocomplement to the constants,

(b) $U_T \otimes U_{T^\alpha,\chi}$ has a simple spectrum if $\chi \neq 0$,

(c) $U_T \otimes U_{T^\alpha,\chi}$ and $U_T \otimes U_{T^\alpha,\xi}$ are unitarily equivalent if χ and ξ belong to the same $\widehat{v}$-orbit,

(d) the measures of maximal spectral type of $U_T \otimes U_{T^\alpha,\chi}$ and $U_T \otimes U_{T^\alpha,\xi}$ are mutually singular if χ and ξ are not on the same $\widehat{v}$-orbit.

It follows from Lemma 5.3(ii) that $\mathrm{WC}(U_T)$ contains operators $0.5(I+U_T(-\xi_1))$, $0.5(I+U_T(-2\xi_1))$, $0.5(I+U_T(-\xi_2))$ and $0.5(I+U_T(\xi_1-\xi_2))$. Therefore we deduce (a) from Lemma 5.1.

Fix a nontrivial $\chi \in \widehat{K}$. The unitary representation $U_{T^\alpha,\chi}$ has a simple spectrum (see the proof of Theorem 3.1). Moreover,

$$U_{T^\alpha,\chi}(h_n) \to 0.5(I + U_{T^\alpha,\chi}(-\xi_1)), \quad U_T(h_n) \to 0.5(I + U_T(-\xi_1))$$
$$\text{as } \mathcal{M}_{0,1} - 1 \ni n \to \infty \quad \text{and}$$
$$U_{T^\alpha,\chi}(h_n) \to 0.5(I + U_{T^\alpha,\chi}(-\xi_2)), \quad U_T(h_n) \to 0.5(I + U_T(-\xi_2))$$
$$\text{as } \mathcal{M}_{0,2} - 1 \ni n \to \infty.$$

by Lemma 5.3(ii). Since χ is nontrivial, it follows from Algebraic Lemma 1.1 that there is $a \in \mathcal{K}$ with $l_\chi(a) \neq 1$. Again by Lemma 5.3(ii),

$$U_{T^\alpha,\chi}(h_n) \to 0.5(l_\chi(a)I + U_{T^\alpha,\chi}(-\xi_1)), \quad U_T(h_n) \to 0.5(I + U_T(-\xi_1))$$
$$\text{as } \mathcal{M}_{a,1} - 1 \ni n \to \infty \quad \text{and}$$
$$U_{T^\alpha,\chi}(h_n) \to 0.5(l_\chi(a)I + U_{T^\alpha,\chi}(-\xi_2)), \quad U_T(h_n) \to 0.5(I + U_T(-\xi_2))$$
$$\text{as } \mathcal{M}_{a,2} - 1 \ni n \to \infty.$$

Therefore Lemma 5.2 implies (b).

As in the proof of Theorem 4.3 we can define a transformation $S_{\bar{z}}$ of (X,μ) by the formula (3-1). Then $S_{\bar{z}} \in C(T)$. It follows from (A1), (B1) and the definition of C_{n+1} that (3-2) is satisfied. Hence by Lemma 3.3, the cocycle $\alpha\circ S_{\bar{z}}$ is cohomologous to $v\circ\alpha$. Therefore the unitary representations $U_{T,\chi}$ and $U_{T,\xi}$ are unitarily equivalent whenever χ and ξ lie on the same orbit of $\widehat{v}$. This yields (c).

To prove (d), we first find $a \in \mathcal{G}$ such that $l_\chi(a) \neq l_\xi(a)$ (see claim (ii) of Algebraic Lemma). It follows from Lemma 5.3(i) that

$$U_T(h_n)\otimes U_{T^\alpha,\chi}(h_n) \to l_\chi(a)I \quad\text{and}\quad U_T(h_n)\otimes U_{T^\alpha,\xi}(h_n) \to l_\xi(a)I$$

as $\mathcal{N}_a - 1 \ni n \to \infty$. This implies (d).

Finally, since $\mathcal{M}(T\times T^{\alpha,H}) \not\ni 1$, it follows that $T\times T^{\alpha,H}$ is weakly mixing. $\square$

Remark 5.5. It follows from Lemma 5.3(ii) that $U_{T^{\alpha,H}}(h_n) \to 0.5(I + U_{T^{\alpha,H}}(-\xi_i))$ as $\mathcal{M}_{0,i} - 1 \ni n \to \infty$, $j = 1,2,3$. As in Proposition 4.3 we can deduce from this fact that $\mathcal{M}(T^{\alpha,H}) = \mathcal{M}(T_t^{\alpha,H})$ for each $t \neq 0$.

6. Spectral multiplicities for ergodic actions of other groups

The main result of the paper extends partly to actions of some other locally compact second countable Abelian groups G. If G is compact then each ergodic action T of G has a pure point spectrum and $\mathcal{M}(T) = \{1\}$. Therefore from now on we assume that G is non-compact.

Corollary 6.1. *Let G be a torsion free discrete countable Abelian group and let E be a subset of $\mathbb{N}$ such that $E\cap\{1,2\}\neq\emptyset$. Then there is a weakly mixing free action S of G such that $\mathcal{M}(S) = E$.*

Proof. In the case when $G = \mathbb{Z}$ see [Da3] and references therein. Consider now the case when $G \neq \mathbb{Z}$. Then there is an embedding ϕ of G into $\mathbb{R}$ such that the subgroup $\phi(\mathbb{R})$ is dense in $\mathbb{R}$. Indeed, it is well known that G embeds into $\mathbb{Q}^{\mathbb{N}}$. In turn, the later group obviously embeds into $\mathbb{R}$. It remains to note that if an infinite subgroup of $\mathbb{R}$ is not isomorphic to $\mathbb{Z}$ then it is dense in $\mathbb{R}$.

By Theorem 0.1, there is a weakly mixing action T of $\mathbb{R}$ such that $\mathcal{M}(T) = E$. Then the composition $T\circ\phi = (T_{\phi(g)})_{g\in G}$ is a weakly mixing action of G with $\mathcal{M}(T\circ\phi) = \mathcal{M}(T) = E$. $\square$

The first claim of the following lemma is, in fact, a slight generalization of Theorem 4.1. If we replace (relax) "weak mixing" in its statement with "ergodic" then it follows from Theorem 4.1 via Proposition 1.1.

Lemma 6.2. *Let A be a compact second countable Abelian group. Let E be a subset of $\mathbb{N}$ with $1\in E$.*

(i) *There is a weakly mixing free action W of $\mathbb{R}\times A$ such that $\mathcal{M}(W) = E$.*
(ii) *For each torsion free discrete countable Abelian group G, there is a weakly mixing free action W of $G\times A$ such that $\mathcal{M}(W) = E$.*

Proof. (i) Let the objects K, H, v be defined exactly as in Section 4. We now set $K' := K\times A$, $H' := H\times\{0\}$ and $v' := v\times \mathrm{Id}$. It is straightforward that

$$L(\widehat{K'}, \widehat{K'/H'}, \widehat{v'}) = L(\widehat{K}, \widehat{K/H}, \widehat{v}) = E. \tag{6-1}$$

Moreover, the subgroup of v'-periodic points is countable and dense in K'. We now construct the skew product flow $T^{\alpha',H'}$ in the same way as in Section 4 but with K', H', v' instead of K, H, v. We note that $T^{\alpha',H'}$ acts on the space $(Y, \nu) := (X \times K/H \times A, \mu \times \lambda_{K/H} \times \lambda_A)$. Denote by $W = (W_{t,a})_{(t,a)\in\mathbb{R}\times A}$ the action of the product group $\mathbb{R} \times A$ on (Y, ν) generated by $T^{\alpha',H'}$ and the action of A by rotations along the third coordinate. Then W is free. Since $T^{\alpha',H'}$ is weakly mixing, so is W. Denote by U_W the corresponding Koopman unitary representation of $\mathbb{R} \times A$ in $L^2_0(Y, \nu)$. We have a decomposition

$$L^2(Y, \nu) = \bigoplus_{\chi\in\widehat{K/H}, \eta\in\widehat{A}} \mathcal{H}_{\chi,\eta},$$

where $\mathcal{H}_{\chi,\eta} := L^2(X, \mu) \otimes \chi \otimes \eta$. We know from Section 4 that $\mathcal{H}_{\chi,\eta}$ is a $U_{T^{\alpha',H'}}$-cyclic subspace for each pair χ, η. It is also a U_W-cyclic subspace. The unitary operator $U_W(0, a)$ acts on $\mathcal{H}_{\chi,\eta}$ by multiplying on $\eta(a)$. Hence if $\sigma_{\chi,\eta}$ is a measure of maximal spectral type of $U_{T^{\alpha',H'}} \restriction \mathcal{H}_{\chi,\eta}$ then the measure $\sigma_{\chi,\eta} \times \delta_\eta$ on $\widehat{R} \times \widehat{A}$ is a measure of maximal spectral type of $U_W \restriction \mathcal{H}_{\chi,\eta}$. As was shown in Section 4, if (χ, η) and (χ', η') belong to different v'-orbits then $\sigma_{\chi,\eta} \perp \sigma_{\chi',\eta'}$. It follows that $\sigma_{\chi,\eta} \times \delta_\eta \perp \sigma_{\chi',\eta'} \times \delta_{\eta'}$. On the other hand, if (χ, η) and (χ', η') belong to the same v'-orbit then $\sigma_{\chi,\eta} \sim \sigma_{\chi',\eta'}$. Moreover, $\eta = \eta'$ by the definition of v'. Hence $\sigma_{\chi,\eta} \times \delta_\eta \sim \sigma_{\chi',\eta'} \times \delta_{\eta'}$. These facts plus (6-1) imply that $\mathcal{M}(W) = E$.

(ii) Consider two cases. If G is not $\mathbb{Z}$ then (ii) follows from (i) in the very same way as Corollary 6.1 follows from Theorem 0.1. If G is $\mathbb{Z}$ then we need to modify the proof of the main result from [Da3] (only the case when $E \ni 1$) in the very same way as we modified the proof of Theorem 4.1 in (i). □

Let T_1 and T_2 be probability preserving ergodic actions of locally compact second countable Abelian groups G_1 and G_2 respectively. Let $T_1 \otimes T_2$ stand for the product action $(g_1, g_2) \mapsto T_1(g_1) \times T_2(g_2)$ of the product group $G_1 \times G_2$. It is easy to see that

(•) if T_1 has a simple spectrum then $\mathcal{M}(T_1 \otimes T_2) = \mathcal{M}(T_2) \cup \{1\}$.

As far as we know, this fact was first used in [Fi] for $\mathbb{Z}^2$-actions.

Corollary 6.3. *Let $E \ni 1$. If one of the following conditions is satisfied:*

(i) *G contains a closed one-parameter subgroup,*

(ii) *$G = D \times F$, where D is a torsion free discrete countable Abelian group and F is a locally compact second countable Abelian group,*

then there is a free weakly mixing action T of G such that $\mathcal{M}(T) = E$.

Proof. (i) It follows from [HR, Theorem 24.30] that G is topologically isomorphic to a product $\mathbb{R} \times G'$, where G' is a locally compact Abelian group.

Suppose first that G' is non-compact. We now claim that there is a weakly mixing free G'-action with a simple spectrum. To prove this claim we need several standard auxiliary facts which we state here without proof.

- Let $\mathcal{A}$ be the set of all G'-actions on a standard probability space (X, μ). A G'-action is considered as a continuous map from G' to the Polish (in the weak topology) group of all transformations of (X, μ). Then $\mathcal{A}$ endowed with the topology of uniform convergence on the compact subsets in G' is Polish.

– The conjugacy class of every free G'-action is dense in $\mathcal{A}$.
– The subset of all weakly mixing G'-actions and the subset of all G'-actions with a simple spectrum are both G_δ in $\mathcal{A}$.
– There is a weakly mixing free G'-action and there is a free G'-action with a simple spectrum.

The claim follows from them via a *generic argument*. Now we deduce the assertion of the Corollary 6.3 from Theorem 0.1 and ($\bullet$).

Consider now the second case when G' is compact. Then the assertion of the Corollary 6.3 follows from Lemma 6.2(i).

(ii) is proved in a similar way by replacing the references to Theorem 0.1 and Lemma 6.2(i) with references to Corollary 6.1 and Lemma 6.2(ii) respectively. $\square$

We note that if G is connected then (i) is satisfied. If G has no non-trivial compact subgroups then one of the conditions of Corollary 6.3 is satisfied.

We claim that Theorem 0.1, the case $2 \in E$, holds true if we replace $\mathbb{R}$-actions with actions of groups G which are isomorphic to the product of $\mathbb{R}^d$ with torsion free discrete Abelian groups. However to prove this fact one has to pass all the way of Section 5 by adjusting all the arguments from there to the case of $\mathbb{R}^d$-actions. We leave this routine to the reader.

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